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# Constants of motion and the conformal anti - de Sitter algebra in (2+1)-Dimensional Gravity

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Constants of motion are calculated for 2+1 dimensional gravity with topology $\mathbb{R} \times T^2$ and negative cosmological constant. Certain linear combinations of them satisfy the anti - de Sitter algebra $\mathrm{so}(2,2)$ in either ADM or holonomy variables. Quantisation is straightforward in terms of the holonomy parameters. On inclusion of the Hamiltonian three new global constants are derived and the quantum algebra extends to that of the conformal algebra $\mathrm{so}(2,3)$. The modular group appears as a discrete subgroup of the conformal group. Its quantum action is generated by these conserved quantities.

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## I. INTRODUCTION

The Einstein equations for pure 2+1 gravity (with or without a cosmological constant), formulated for spacetimes having a compact Cauchy surface, can be reduced (à la ADM) to a finite dimensional, time dependent Hamiltonian system defined on the cotangent bundle of the Teichmüller space of the chosen (Cauchy) surface [1]. This ADM-reduced system admits sufficiently many independent conserved quantities (traces of holonomies) that one can, in principle at least, determine the evolution of the Teichmüller parameters and their conjugate momenta (i.e., the reduced ADM variables) by simply evaluating the conserved quantities in terms of them and the chosen ADM time coordinate, setting the resultant expressions equal to certain constants fixed by the desired initial conditions, and solving algebraically for the ADM variables as functions of the chosen constants and time. In other words, merely evaluating the conserved quantities in terms of the ADM phase space variables and time gives the solution of Hamilton's equations implicitly. This procedure can be carried out explicitly for the case of Cauchy surfaces diffeomorphic to the torus (the spherical case being essentially trivial and the higher genus case nearly intractable in practice).

The reduced ADM dynamics for the torus case can be quantised in a straightforward way by converting the Hamiltonian to a positive self-adjoint operator (the square root of a Laplace-Beltrami operator) acting on square-integrable functions on Teichmüller space (or, more precisely, on moduli space). The associated quantum dynamics has been studied in some detail by Puzio for the vacuum case [2]. One can ask, however, whether there is another approach to solving the quantum dynamics modelled on the classical technique outlined above. Could one define quantum analogues of the classical conserved quantities, set these equal to certain constant operators having appropriate commutation relations and solve the resulting formulae for the ADM operators (representing the quantised Teichmüller variables and their momenta) in terms of a set of fixed operators and time? This would amount to solving the Heisenberg equations of motion by a technique which closely parallels the solution of the classical Hamilton equations descibed above. Some guidance for doing this is provided by the classical Poisson bracket algebra of the conserved quantities which, presumably, one wishes to implement quantum mechanically if possible. In this paper we study this algebra and its possible quantum implementations from several points of view which are not necessarily

equivalent at the quantum level.

This quantum mechanical implementation of the classical algebra is indeed successful provided we first express the conserved quantities in terms of certain holonomy parameters which were first introduced in [3]. The conserved quantities then take the form of purely quadratic expressions for which quantisation is straightforward. Having done this we then proceed to study the quantum action of the modular group upon the chosen representation of the conserved quantities. We also present a time dependent transformation relating the quantised holonomy parameters and the moduli and their momenta (i.e. the ADM variables) that was first discovered, for $\Lambda = 0$, in [4] and extended to $\Lambda < 0$ in [5]. However we should mention that the quantum dynamics may not be unitarily equivalent to that involving the (square root) Hamiltonian studied by Puzio [2]. At least we do not know of such a unitary correspondence and suspect that it may not exist since otherwise the simple formula given in [5] would encode within it the details of the spectrum of the Laplacian on moduli space. Thus, even though we present a solution of the Heisenberg equations of motion in terms of certain constant operators which are simply related to the quantum analogues of the conserved quantities discussed above, we do not claim that this is unitarily related to the Schrödinger problem posed by Puzio [2] whose solution therefore remains open.

The ADM approach to quantisation has the advantage of focusing on dynamics and it has a relatively clear correspondence to the classical theory. Its main disadvantages are that it requires the *a priori* choice of a time gauge (presumably breaking "general covariance" at the quantum level) and that it leads to nearly intractable dynamics (even classically) for the higher genus cases. An alternative approach to quantisation developed by Nelson and Regge [3,6-7] has the great advantages that it does not require the *a priori* choice of a gauge and that it remains tractable for the higher genus problems. Its principal disadvantage is perhaps that it does not (as yet) deal with dynamical questions and thus is more difficult to connect (in the correspondence limit) to the classical picture of space-time as "space evolving in time". We hope that our present approach, however incomplete, may eventually lead to a reconciliation of these seemingly disparate approaches to quantum gravity in 2+1 dimensions.

The structure of the paper is as follows. In Section II we derive a set of constants of the motion and express them in terms of both the ADM variables and the global holonomy parameters. In Section III we show that certain com-

binations of these constants satisfy the classical anti - de Sitter Poisson bracket algebra. In Section IV we discuss the quantisation of this algebra and show how it is straightforward in terms of the holonomy parameters. In Section V the Hamiltonian is included in the algebra leading to three new global, quantum, constants of the motion. This extended algebra is isomorphic to the algebra of the conformal group SO$(2,3)$. In Section VI the action of the modular group is calculated and shown to be generated by precisely the (quantum) constants of the motion which generate a discrete subgroup of the conformal group. Our results are summarised in Section VII.

## II. CONSTANTS OF THE MOTION

As discussed in [1], one can show that the vacuum spacetimes on $\mathbb{R} \times T^2$ having negative cosmological constant $\Lambda$ and admitting a Cauchy surface of constant mean curvature $\tau$ are all spatially homogeneous. In suitable coordinates their metrics can be written in the form

$$ds^2 = -N(t)^2(dt)^2 + e^{2\mu(t)}(dx^1)^2 + e^{2\nu(t)}(dx^2 + \beta(t)dx^1)^2 \tag{2.1}$$

where $x^1$ and $x^2$ are each periodic (with period $= 1$ for convenience) on $S^1$.

As in [8] we introduce the orthonormal frame

$$\begin{aligned} e^{(0)} &= N(t)dt, \quad e^{(1)} = e^{\mu(t)}dx^1, \\ e^{(2)} &= e^{\nu(t)}(dx^2 + \beta(t)dx^1) \end{aligned} \tag{2.2}$$

and compute the connection one-forms $\omega_{(a)(b)} = \omega_{(a)(b)\mu}dx^\mu$

$$\begin{aligned} \omega_{(1)(0)} &= -\omega_{(0)(1)} = A(t)dx^1 + C(t)dx^2, \\ \omega_{(2)(0)} &= -\omega_{(0)(2)} = B(t)dx^1 + D(t)dx^2, \\ \omega_{(1)(2)} &= -\omega_{(2)(1)} = \frac{1}{2}e^{\nu(t)-\mu(t)}\beta_{,t}\ dt, \end{aligned} \tag{2.3}$$

where

$$
\begin{aligned}
A(t) &= \frac{1}{N(t)}(e^{\mu}\mu_{,t} + \frac{1}{2}\ e^{2\nu-\mu}\beta\beta_{,t}) \\
B(t) &= \frac{1}{N(t)}(\beta e^{\nu}\nu_{,t} + \frac{1}{2}\ e^{\nu}\beta_{,t}) \\
C(t) &= \frac{1}{N(t)}(\frac{1}{2}\ e^{2\nu-\mu}\beta_{,t}) \\
D(t) &= \frac{1}{N(t)}e^{\nu}\nu_{,t}.
\end{aligned}
\tag{2.4}
$$

In terms of these quantities we introduce the duals

$$
\omega^{(a)} = \frac{1}{2}\ \epsilon^{abc}\omega_{(b)(c)} \tag{2.5}
$$

and a pair of "shifted connections" $\lambda^{\pm(a)}$ defined by

$$
\lambda^{\pm(a)} = \omega^{(a)} \pm \sqrt{-\Lambda}\ e^{(a)}. \tag{2.6}
$$

From [9] we know that the traces of the SO$(1,2)$ holonomies of $\lambda^{+}$ and $\lambda^{-}$, defined for arbitrary closed loops in these spacetimes, are absolutely conserved quantities (i.e., they are gauge invariant and invariant under non-singular deformations of the loops within the vacuum spacetimes).

As in [8] we compute this pair of traces for 3-different classes of loops represented respectively by the "a-loops" having $x^2 = constant$, the "b-loops" having $x^1 = constant$ and the "twisting loops" having $x^1 = \frac{p}{q}\ x^2$. The twisting loops do not yield new independent conserved quantities but rather give certain functions of those coming from the a- and b-loops.

Using a convenient representation of SO$(1,2)$ as in [8] and [9] one extracts from the traces the following conserved quantities

$$
\begin{aligned}
C_1^{\pm} &= (B \pm \sqrt{-\Lambda}e^{\mu})^2 + (-A \pm \sqrt{-\Lambda}\ e^{\nu}\beta)^2, \\
C_2^{\pm} &= D^2 + (-C \pm \sqrt{-\Lambda}e^{\nu})^2,
\end{aligned}
\tag{2.7}
$$

(which come from the a- and b-loops respectively) and

$$
C_3^{\pm} = (-A \pm \sqrt{-\Lambda}e^{\nu}\beta)(-C \pm \sqrt{-\Lambda}\ e^{\nu}) + D(B \pm \sqrt{-\Lambda}e^{\mu}) \tag{2.8}
$$

(which come from the twisting loops).

For the sake of easy comparison with earlier work on the $\Lambda = 0$ problem we decompose the above expressions into the following equivalent set of conserved quantities

$$\begin{aligned}
C_1 &= \frac{(C_1^+ + C_1^-)}{2} = A^2 + B^2 - \Lambda(e^{2\mu} + e^{2\nu}\beta^2) \\
C_2 &= \frac{(C_2^+ + C_2^-)}{2} = C^2 + D^2 - \Lambda e^{2\nu} \\
C_3 &= \frac{(C_3^+ + C_3^-)}{2} = AC + BD - \Lambda e^{2\nu}\beta \\
C_4 &= \frac{(C_1^+ - C_1^-)}{4\sqrt{-\Lambda}} = Be^{\mu} - Ae^{\nu}\beta \\
C_5 &= -\frac{(C_2^+ - C_2^-)}{4\sqrt{-\Lambda}} = e^{\nu}C \\
C_6 &= \frac{(C_3^+ - C_3^-)}{2\sqrt{-\Lambda}} = e^{\mu}D - e^{\nu}\ \beta\ C - e^{\nu}A
\end{aligned} \tag{2.9}$$

In the limit that $\Lambda \to 0$ these quantities reduce to those defined in [8].

### A. ADM variables

As in [8] we define new canonical coordinates $\{q^i\}$ by

$$q^1 = \nu - \mu, q^2 = \beta, q^3 = \nu + \mu \tag{2.10}$$

and introduce their conjugate momenta $\{p_i\}$ such that

$$\sum_i p_i q^i_{,t} = \sum \pi^{ab} g_{ab,t} \tag{2.11}$$

where $\{g_{ab}, \pi^{ab}\}$ are the usual Arnowitt, Deser and Misner (ADM) canonical variables for the metric (2.1). With these definitions $q^1$ and $q^2$ parametrize the conformal metric

$$h_{ab} = \frac{g_{ab}}{\sqrt{^{(2)}g}} \tag{2.12}$$

while $q^3$ parametrizes the spatial volume element, $e^{q^3} = \sqrt{^{(2)}g}$ and the mean curvature $\tau$ is given by

$$\tau = \frac{p_3}{e^{q^3}} \tag{2.13}$$

By virtue of spatial homogeneity the momentum constraints are satisfied identically while the Hamiltonian constraint now takes the form

$$\mathcal{H} = \frac{1}{e^{q^3}}(\frac{1}{2}p_1^2 + \frac{1}{2}\ e^{-2q^1}p_2^2 - \frac{1}{2}p_3^2) + 2\Lambda e^{q^3}. \tag{2.14}$$

The ADM super Hamiltonian is thus

$$H_{super} = \int_{T^2} N\mathcal{H}d^2x = N\mathcal{H} \tag{2.15}$$

from which one derives the relations

$$\begin{aligned} \dot{q}^1 &= \frac{N}{e^{q^3}}p_1,\ \dot{q}^2 = \frac{N}{e^{q^3}}\ e^{-2q^1}p_2, \\ \dot{q}^3 &= -\frac{N}{e^{q^3}}p_3. \end{aligned} \tag{2.16}$$

Using (2.4) and (2.10-11) we can easily express the conserved quantities $C_1 - C_6$ in terms of the canonical variables $\{q^i, p_i\}$. First however we wish to ADM-reduce the dynamics by choosing the York time coordinate condition

$$t = \tau = \frac{p_3}{e^{q^3}} \tag{2.17}$$

and by solving the constraint $\mathcal{H} = 0$ (2.14) for $e^{q^3}$ which serves to eliminate the pair $\{q^3, p_3\}$ from the dynamical equations. With this choice of gauge the reduced, ADM, Hamiltonian is just the spatial volume

$$H_{ADM} = \int_{T^2} d^2x\sqrt{^{(2)}g} = e^{q^3} = \frac{1}{\sqrt{\tau^2 - 4\Lambda}}\ \bar{H} \tag{2.18}$$

where

$$\bar{H} = \sqrt{p_1^2 + e^{-2q^1}p_2^2} \tag{2.19}$$

is explicitly time independent.

Thus, as expected since volume is not conserved, the ADM Hamiltonian (2.18) is explicitly time dependent. Here however the time dependence resides in a multiplicative factor which we can eliminate by simply changing the time variable from York time $\tau$ to a new time $t'$ defined by

$$\tau = \sqrt{-4\Lambda} \sinh t' \tag{2.20}$$

With this choice the reduced Hamilton equations are

$$\frac{dq^i}{dt'} = \frac{\partial \bar{H}}{\partial p_i} , \quad \frac{dp_i}{dt'} = -\frac{\partial \bar{H}}{\partial q^i} \tag{2.21}$$

with $\bar{H}$ given by (2.19).

In terms of the foregoing definitions, the conserved quantities $C_1 - C_6$ now take the form

$$\begin{aligned}
C_1 &= \frac{1}{2}\, e^{-q^1} \tau \left\{ (\sqrt{1 - \frac{4\Lambda}{\tau^2}}\bar{H} - p_1)(1 + (q^2)^2 e^{2q^1}) - 2(q^2 p_2 - p_1) \right\} , \\
C_2 &= \frac{1}{2}\, e^{q^1} \tau \left\{ \sqrt{1 - \frac{4\Lambda}{\tau^2}}\bar{H} - p_1 \right\} , \\
C_3 &= \frac{1}{2}\, e^{q^1} \tau \left\{ q^2 (\sqrt{1 - \frac{4\Lambda}{\tau^2}}\bar{H} - p_1) - p_2\, e^{-2q^1} \right\} , \\
C_4 &= \frac{1}{2} \left\{ p_2\, e^{-2q^1} + 2q^2 p_1 - p_2 (q^2)^2 \right\} , \\
C_5 &= \frac{1}{2}\, p_2 \\
C_6 &= p_1 - q^2 p_2
\end{aligned} \tag{2.22}$$

and reduce to those found previously [8] when $\Lambda \to 0$.

While $C_1 - C_6$ are conserved quantities by construction one could verify this fact directly using the reduced Hamilton equations

$$\frac{dq^i}{dt'} = \frac{\partial \bar{H}}{\partial p_i} , \quad \frac{dp_i}{dt'} = -\frac{\partial \bar{H}}{\partial q^i}, \quad i = 1, 2. \tag{2.23}$$

We shall verify this in a different way below but mention here that one can simply use the constancy of the $C_i$'s to solve Hamilton's equations by purely algebraic means. One can simply choose four of the independent $C_i$'s and solve them for the four canonical variables $\{q^i, p_i\}$ in terms of $\tau = \sqrt{-4\Lambda} \sinh t'$ and the four independent constants. That this produces the solution to Hamilton's equations is equivalent to the fact that four of the $C_i$'s are functionally independent constants of the motion.

## B. Holonomy parameters

That the $C_1 - C_6$ are time independent can be seen alternatively by reexpressing these quantities in terms of the global, time independent parameters $r^{\pm}_{1,2}$ of the traces of the SL(2,IR) holonomies (Wilson loops) [3,5], expressed as

$$\begin{aligned} R^{\pm}_1 &= \cosh \frac{r^{\pm}_1}{2} \\ R^{\pm}_2 &= \cosh \frac{r^{\pm}_2}{2} \end{aligned} \tag{2.24}$$

where the subscripts 1 and 2 in (2.24) refer to two intersecting paths $\gamma_1, \gamma_2$ on $T^2$ ( the "a-loops" and "b-loops" of Section IIA, respectively) with intersection number +1. (A third holonomy, $R^{\pm}_{12} = \cosh{(r^{\pm}_1 + r^{\pm}_2)}/2$, corresponds to the path $\gamma_1 \cdot \gamma_2$, the "twisting loops", which has intersection number $-1$ with $\gamma_1$ and +1 with $\gamma_2$.) In (2.24) the $\pm$ refer to the two copies of SL(2,IR) which appear in the decomposition of the spinor group of $SO(2,2)$ as a tensor product SL(2,IR) $\otimes$ SL(2,IR).

The holonomies (2.24) are the normalised traces of the hyperbolic-hyperbolic representation of SL(2,IR) (here hyperbolic means that the normalised trace is $> 1$). This hyperbolic-hyperbolic representation is necessary for the toroidal slices to be spacelike [10]. They completely solve all constraints in the alternative first order, formalism, and can be calculated directly from the classical solutions (2.2-3), or from the "shifted connections" (2.6) in the time gauge $N = 1, N^i = 0$. This gauge is equivalent, for the topology $\mathrm{IR} \times T^2$, to the York gauge of Section IIA. Explicitly,

$$(r^{\pm}_{1,2})^2 = \Delta^{\pm}_{1,2}{}^a \ \Delta^{\pm}_{1,2}{}^b \ \eta_{ab}, \quad \eta_{ab} = diag(-1,1,1)$$

with

$$\Delta^{\pm}_{1,2}{}^a = \int_{\gamma_1,\gamma_2} \lambda^{\pm(a)} \tag{2.25}$$

and $\lambda^{\pm(a)}$ is given by (2.6). Moreover, the holonomies (2.24) satisfy the nonlinear classical Poisson bracket algebra [4,6]

$$\begin{aligned} \{R^{\pm}_1, R^{\pm}_2\} &= \mp \frac{1}{4\alpha}(R^{\pm}_{12} - R^{\pm}_1 R^{\pm}_2) \\ \{R^{+}_1, R^{-}_2\} &= 0, \end{aligned} \tag{2.26}$$

where

$$\alpha = \frac{1}{\sqrt{-\Lambda}} > 0.$$

The algebra (2.26) is obtained by integration of the Poisson brackets

$$\{e_i^{(a)}(\mathbf{x}), \omega_{j(b)(c)}(\mathbf{y})\} = -\frac{1}{2}\epsilon_{ij}\epsilon^a{}_{bc}\delta^2(\mathbf{x}-\mathbf{y}), \quad i,j=1,2, \epsilon_{12}=1 \tag{2.27}$$

along $\gamma_1$ and $\gamma_2$ [3]. When the traces $R^{\pm}_{1,2}$ are represented as in (2.24) it follows that the parameters $r^{\pm}_{1,2}$ satisfy the classical Poisson brackets

$$\{r_1^{\pm}, r_2^{\pm}\} = \mp\frac{1}{\alpha}, \qquad \{r^+, r^-\} = 0 \tag{2.28}$$

The four real parameters $r_1^{\pm}, r_2^{\pm}$ appearing in (2.24) are arbitrary, but in [5] it was shown that they are related, through a time-dependent canonical transformation, to the components of the moduli $m = m_1 + im_2$ and their momenta $\pi = \pi^1 + i\pi^2$ as follows*.

$$m = \left(r_1^- e^{it/\alpha} + r_1^+ e^{-it/\alpha}\right)\left(r_2^- e^{it/\alpha} + r_2^+ e^{-it/\alpha}\right)^{-1} \tag{2.29}$$

$$\pi = -\frac{i\alpha}{2\sin\frac{2t}{\alpha}}\left(r_2^+ e^{it/\alpha} + r_2^- e^{-it/\alpha}\right)^2 \tag{2.30}$$

where $m$ and $\pi$ are related to the ADM variables $q^1, q^2, p_1, p_2$ of Section IIA by

$$m_1 = q^2, m_2 = e^{-q^1}, \pi^1 = p_2, \pi^2 = -p_1 e^{q^1} \tag{2.31}$$

The parameter $t$ appearing in (2.29-30) is related to the extrinsic curvature $\tau$ by

$$\tau = -\dot{q}^3 = -\frac{2}{\alpha}\cot\frac{2t}{\alpha} \tag{2.32}$$

with $\tau$ monotonic in the range $t \,\epsilon\, (0, \frac{\pi\alpha}{2})$.

Since the $r_1^{\pm}, r_2^{\pm}$ are arbitrary the moduli and momenta can have arbitrary initial data $m(t_0), p(t_0)$ at some initial time $t_0$.

In [5] it was shown that the moduli $m_1, m_2$ with $m_2 > 0$ lie on a circle

$$(m_1 - c)^2 + m_2^2 = |m - c|^2 = R^2. \tag{2.33}$$

* In [5] $p = p^1 + ip^2$ was used to denote the complex moduli momenta $\pi$. Here $p_1$ and $p_2$ denote the ADM momenta of Section IIA.

It follows that the centre of this circle given by $m_2 = 0, m_1 = c.$ and its radius $R$ can be expressed in terms of the constants of Section IIA by differentiating (2.33) with respect to $t$ and using (2.16), (2.22) and (2.31)

$$c = -\frac{C_6}{2C_5}, \qquad R^2 = \frac{C_6^2 + 4C_4C_5}{4C_5^2}. \tag{2.34}$$

The Poisson brackets (2.28) of the parameters $r^{\pm}_{1,2}$ can be used to calculate those of the moduli and their momenta. From (2.29-30) we find

$$\{\bar{m}, \pi\} = \{m, \bar{\pi}\} = -2, \qquad \{m, \pi\} = \{\bar{m}, \bar{\pi}\} = 0 \tag{2.35}$$

The ADM Hamiltonian (2.18) now takes the form, using (2.29-31)

$$H = g^{1/2} = \frac{\alpha^2}{4} \sin\frac{2t}{\alpha}(r_1^- r_2^+ - r_1^+ r_2^-) = \frac{\alpha}{2\sqrt{\tau^2 - 4\Lambda}}(r_1^- r_2^+ - r_1^+ r_2^-) \tag{2.36}$$

and generates the $\tau$ development of the modulus (2.29) and momentum (2.30) through

$$\frac{d\pi}{d\tau} = \{\pi, H\}, \qquad \frac{dm}{d\tau} = \{m, H\} \tag{2.37}$$

Alternatively, the Hamiltonian

$$H' = \frac{d\tau}{dt} H = \frac{4}{\alpha^2} \csc^2 \frac{2t}{\alpha} H$$

generates evolution in coordinate time $t$ by

$$\frac{d\pi}{dt} = \{\pi, H'\}, \qquad \frac{dm}{dt} = \{m, H'\}$$

Using (2.29-32) one can show that in terms of the holonomy parameters $r^{\pm}_{1,2}$ the constants of the motion $C_1 - C_6$ (2.22) are particularly simple

$$C_1 = \frac{1}{2}((r_1^+)^2 + (r_1^-)^2)$$

$$C_2 = \frac{1}{2}((r_2^+)^2 + (r_2^-)^2)$$

$$C_3 = \frac{1}{2}(r_1^+ r_2^+ + r_1^- r_2^-)$$

$$C_4 = \frac{\alpha}{4}((r_1^-)^2 - (r_1^+)^2)$$

$$C_5 = \frac{\alpha}{4}((r_2^+)^2 - (r_2^-)^2)$$

$$C_6 = \frac{\alpha}{4}(r_1^- r_2^- - r_1^+ r_2^+) \tag{2.38}$$

or alternatively, from (2.9)

$$C_1^{\pm} = (r_1^{\mp})^2$$

$$C_2^{\pm} = (r_2^{\mp})^2$$

$$C_3^{\pm} = r_1^{\mp} r_2^{\mp} \tag{2.39}$$

and they are evidently time independent. It can easily be checked from (2.38) that the constants $C_1 - C_6$ are not all independent. They satisfy

$$C_2 C_4 - C_1 C_5 - C_3 C_6 = 0 \tag{2.40}$$

and the time independent part $\bar{H}$ (2.19) of the ADM Hamiltonian (2.18), where

$$\bar{H} = \frac{\alpha}{2}(r_1^- r_2^+ - r_1^+ r_2^-) \tag{2.41}$$

is expressed by either

$$\Lambda \bar{H}^2 = (C_3)^2 - C_1 C_2 \quad or \quad \bar{H}^2 = (C_6)^2 + 4 C_4 C_5 \tag{2.42}$$

## III. THE ANTI-DE SITTER ALGEBRA

In [8] it was shown that suitable combinations of these six constants of the motion satisfy the Lie algebra of the Poincaré group*. Here the combinations that satisfy the Lie algebra of the anti-de Sitter group SO$(2, 2)$ are

$$P_0 = -\frac{1}{2}(C_1 + C_2), P_1 = \frac{1}{2}(C_1 - C_2), P_2 = C_3$$

$$J_{12} = C_5 - C_4, J_{02} = C_4 + C_5, J_{01} = -C_6 \tag{3.1}$$

that is

$$\{J_{ab}, J_{cd}\} = \eta_{ac} J_{bd} - \eta_{bc} J_{ad} - \eta_{ad} J_{bc} + \eta_{bd} J_{ac}$$

$$\{P_a, P_b\} = \Lambda J_{ab}$$

* The sign of $P_0$ reported in [8] was incorrect. The version here is the correct one.

$$\{J_{ab}, P_c\} = \eta_{ac}P_b - \eta_{bc}P_a \tag{3.2}$$

where

$$a, b, c = 0, 1, 2, \quad \eta_{ab} = diag(-1, 1, 1)$$

satisfying

$$P_a J_{bc} \epsilon^{abc} = 0, \quad \epsilon^{012} = -\epsilon_{012} = 1 \tag{3.3}$$

The classical algebra can be checked in either set of variables using the Poisson brackets (2.28) or (2.35) though it is evidently somewhat easier in the holonomy variables.

It is useful to define the generators

$$j_a{}^{\pm} = \frac{1}{2}\epsilon_{abc}J^{bc} \pm \alpha P_a \tag{3.4}$$

with each $\pm$ copy satisfying the Lie algebra of so$(1,2) \approx$ sl(2,$\mathbb{R}$).

$$\{j_a^{\pm}, j_b^{\pm}\} = 2\epsilon_{abc}j^{c\pm}$$

$$\{j_a{}^{+}, j_b{}^{-}\} = 0 \tag{3.5}$$

with

$$j = j_a{}^{+}j^{a+} = j_a{}^{-}j^{a-} = 0 \tag{3.6}$$

Explicitly we have

$$j_0^{\pm} = \mp\frac{\alpha}{2}(C_1^{\mp} + C_2^{\mp}) = \mp\frac{\alpha}{2}((r_1^{\pm})^2 + (r_2^{\pm})^2)$$

$$j_1^{\pm} = \mp\frac{\alpha}{2}(C_2^{\mp} - C_1^{\mp}) = \pm\frac{\alpha}{2}((r_1^{\pm})^2 - (r_2^{\pm})^2)$$

$$j_2^{\pm} = \pm\alpha C_3^{\mp} = \pm\alpha r_1^{\pm}r_2^{\pm} \tag{3.7}$$

Note that $j_a^+$ depends only on the $r^+$'s and $j_a^-$ only on the $r^-$'s. The time independent part $\bar{H}$ (2.41) of the ADM Hamiltonian $H$ (2.18) or (2.36) is assumed to be positive. This guarantees that the imaginary part $m_2$ of the modulus (2.29) is also positive in the range $t \,\epsilon\, (0, \frac{\pi\alpha}{2})$, since, from (2.29)

$$m_2 = e^{-q^1} = \sin\frac{2t}{\alpha}\frac{(r_1^- r_2^+ - r_1^+ r_2^-)}{|r_2^- e^{it/\alpha} + r_2^+ e^{-it/\alpha}|^2}.$$

$\bar{H}$ can also be expressed, from (2.42), (3.1) and (3.7), in terms of the anti-de Sitter and sl(2,$\mathbb{R}$) generators

$$\bar{H}^2 = -\frac{1}{2}J_{ab}J^{ab} = \frac{1}{2}j_a^+ j^{a-} \tag{3.8}$$

$$\Lambda\bar{H}^2 = P_a P^a \tag{3.9}$$

## IV.QUANTUM THEORY

### A. ADM Quantisation

To quantise the ADM-reduced dynamics one can proceed as suggested in [1] and developed in detail in [2] for the vacuum case. Indeed the only essential difference between the reduced Schrödinger dynamics here and that for $\Lambda = 0$ is that the relationship between the time coordinate $t'$ (c.f. (2.20)) and the mean curvature depends upon $\Lambda$ -the reduced Hamiltonian operator is independent of $\Lambda$.

A "choice" which must be made in either case is whether to formulate the reduced quantum mechanics on the full Teichmüller space for the torus (i.e. the 2-dimensional hyperbolic space with global coordinates $q^1, q^2$ and Riemannian metric $(dq^1)^2 + e^{2q^1}(dq^2)^2$ or instead upon the moduli space obtained from quantising the Teichmüller space by the action of the modular group discussed below in Section VI.

Since the latter choice implements invariance of the physical states with respect to "large" diffeomorphisms as well as the small ones which are connected to the identity, it seems to be the natural one to make. The Schrödinger Hamiltonian determined from (2.19) may, as before, be defined as the positive square root of the invariant Laplacian defined on moduli space - a choice which seems to lead to well-defined quantum dynamics for the full physically desirable range of $t'$ [2].

However, as we shall show below in a related context, the conserved quantities $C_1 - C_6$ are not invariant with respect to the transformations generating the modular group (c.f. (6.3)). Thus one does not expect to be able to implement them globally as self-adjoint operators in fully reduced moduli-space quantisation,

a conclusion which seems to have been reached by a more geometrical line of reasoning by Hajicek as well [11].

One could perhaps try to implement the operator analogues of $C_1 - C_6$ on a partially reduced quantisation, working on Teichmüller space instead of moduli space, but unfortunately there is no known ordering of the operator analogues of expressions (2.22) which captures the so$(2,2)$ algebra expected from the classical considerations of Section III. Fortunately, however, this problem disappears when the conserved quantities are instead quantised in terms of the holonomy parameters, as we shall now show.

### B. Holonomy parameter quantisation

The quantisation of the algebra (3.2) is straightforward in terms of the holonomy parameters. Indeed if all the $r^{\pm}_{1,2}$ are promoted to operators $\hat{r}^{\pm}_{1,2}$ satisfying the commutators

$$[\hat{r}^{\pm}_1, \hat{r}^{\pm}_2] = \hat{r}^{\pm}_1\hat{r}^{\pm}_2 - \hat{r}^{\pm}_2\hat{r}^{\pm}_1 = \mp\frac{i\hbar}{\alpha}$$
$$[\hat{r}^{\pm}, \hat{r}^{\mp}] = 0 \tag{4.1}$$

then there are no ordering problems in $\hat{H}$ or $\hat{\tilde{H}}$ (3.8), that is,

$$\hat{H} = \frac{\alpha}{2}\sin\frac{2t}{\alpha}\hat{\tilde{H}}, \quad \hat{\tilde{H}} = \frac{\alpha}{2}(\hat{r}^-_1\hat{r}^+_2 - \hat{r}^+_1\hat{r}^-_2) \tag{4.2}$$

and the moduli and momenta, *ordered as in* (2.29-30) , that is,

$$\hat{m} = \left(\hat{r}^-_1 e^{it/\alpha} + \hat{r}^+_1 e^{-it/\alpha}\right)\left(\hat{r}^-_2 e^{it/\alpha} + \hat{r}^+_2 e^{-it/\alpha}\right)^{-1} \tag{4.2}$$

$$\hat{\pi} = -\frac{i\alpha}{2\sin\frac{2t}{\alpha}}\left(\hat{r}^+_2 e^{it/\alpha} + \hat{r}^-_2 e^{-it/\alpha}\right)^2 \tag{4.4}$$

satisfy

$$[\hat{m}^\dagger, \hat{\pi}] = [\hat{m}, \hat{\pi}^\dagger] = -2i\hbar, \qquad [\hat{m}, \hat{\pi}] = [\hat{m}^\dagger, \hat{\pi}^\dagger] = 0 \tag{4.5}$$

$$[\hat{\pi}, \hat{H}] = i\hbar\frac{d\hat{\pi}}{d\tau}, \qquad [\hat{m}, \hat{H}] = i\hbar\frac{d\hat{m}}{d\tau} \tag{4.6}$$

which follow from the commutators (4.1).

For the sl(2,$\mathbb{R}$) generators (3.7) it is clear that there are no ordering problems in $j^{\pm}_0$ or $j^{\pm}_1$ that is

$$\hat{j}^{\pm}_0 = \mp\frac{\alpha}{2}((\hat{r}^{\pm}_1)^2 + (\hat{r}^{\pm}_2)^2)$$

$$\hat{j}_1^{\pm} = \pm\frac{\alpha}{2}((\hat{r}_1^{\pm})^2 - (\hat{r}_2^{\pm})^2) \tag{4.7}$$

whereas in $j_2^{\pm}$ the symmetric ordering

$$\hat{j}_2^{\pm} = \pm\frac{\alpha}{2}(\hat{r}_1^{\pm}\hat{r}_2^{\pm} + \hat{r}_2^{\pm}\hat{r}_1^{\pm}) \tag{4.8}$$

will give the commutators

$$[\hat{j}_a^{\pm}, \hat{j}_b^{\pm}] = 2i\hbar\epsilon_{abc}\hat{j}^{c\pm}$$

$$[\hat{j}_a^{+}, \hat{j}_b^{-}] = 0 \tag{4.9}$$

The commutator (4.1) defines a spinor norm

$$\epsilon^{AB}\hat{r}_B^{\pm}\hat{r}_A^{\pm} = \hat{r}^{A\pm}\hat{r}_A^{\pm} = [\hat{r}_2^{\pm}, \hat{r}_1^{\pm}] = \pm\frac{i\hbar}{\alpha} \tag{4.10}$$

with $\epsilon^{12} = -\epsilon^{21} = 1$ and $\hat{r}^{A\pm} = \epsilon^{AB}\hat{r}_B^{\pm}$.

The quantum Casimir (3.6) commutes with all the $\hat{j}_a^{\pm}$, and is no longer zero, but $O(\hbar^2)$

$$\hat{j} = \hat{j}_a^{\pm}\hat{j}^{a\pm} = \frac{3\hbar^2}{4} \tag{4.11}$$

Similarly the identity (3.8) acquires $O(\hbar^2)$ corrections

$$\hat{\bar{H}}^2 = \frac{1}{2}\hat{j}_a^{+}\hat{j}^{a-} + \frac{\hbar^2}{2} \tag{4.12}$$

This particular value of the Casimir (4.11) corresponds to a particular discrete representation of $SU(1,1)$ in which $\hat{j}$ and $\hat{j}_0^{\pm}$ are diagonal. This will be discussed elsewhere. Note that the only ordering ambiguity is in $\hat{j}_2{}^{\pm}$ (4.8) but that any other ordering would only produce terms of $O(\hbar^2)$ on the R.H.S. of (4.9).

## V. EXTENDED QUANTUM ALGEBRA

The two quantum so$(1,2)$ algebras introduced in the previous section can be extended to so$(2,3)$ on inclusion of the time independent, constant part $\hat{\bar{H}}$ (4.2) of the ADM Hamiltonian as follows.

The ADM Hamiltonian is not a constant of the motion, in fact classically it represents the surface area of the torus which increases from zero (the initial

singularity) to a maximum and then recollapses. In this section we work only with the time independent part $\hat{\bar{H}}$ (4.2) since the time dependence is simply a multiplicative factor, positive in the range $t\epsilon(0, \frac{\pi\alpha}{2})$. The positivity, classically, of the constant, global part is clearly related to the question of ranges and signs of the classical, constant, holonomy parameters $r_{1,2}{}^{\pm}$. We assumed that $r_1^- r_2^+ - r_1^+ r_2^- > 0$ which guarantees that, classically, $m_2 = e^{-q^1} > 0$. As a quantum mechanical operator $\hat{\bar{H}}$ we cannot guarantee that its spectrum be positive definite without assuming some representation for the operators satisfying the commutators (4.1). This problem is outside the scope of this paper.

It can be checked from (4.2) and (4.7-8) that with

$$\hat{J}_{ab} = -\frac{1}{2}\epsilon_{abc}(\hat{j}^{c+} + \hat{j}^{c-}) \qquad \hat{P}_a = \frac{\hat{j}_a^+ - \hat{j}_a^-}{2\alpha} \tag{5.1}$$

and using the commutators (4.1) it follows that $[\hat{\bar{H}}, \hat{J}_{ab}] = 0$, whereas $[\hat{\bar{H}}, \hat{P}_a] \neq 0$ but instead defines a new constant three-vector $\hat{v}_a$ by

$$[\hat{\bar{H}}, \hat{P}_a] = i\hbar\alpha\ \hat{v}_a, a = 0, 1, 2 \tag{5.2}$$

where

$$\begin{aligned} \hat{v}_0 &= -\frac{\alpha}{2}(\hat{r}_1^+\hat{r}_1^- + \hat{r}_2^+\hat{r}_2^-) \\ \hat{v}_1 &= \frac{\alpha}{2}(\hat{r}_1^+\hat{r}_1^- - \hat{r}_2^+\hat{r}_2^-) \\ \hat{v}_2 &= -\frac{\alpha}{2}(\hat{r}_1^+\hat{r}_2^- + \hat{r}_2^+\hat{r}_1^-) \end{aligned} \tag{5.3}$$

The $\hat{v}_a, \hat{\bar{H}}$ *classically* form a null vector

$$\hat{v}_a\hat{v}^a = \hat{\bar{H}}^2 - \frac{\hbar^2}{2} \tag{5.4}$$

as can be seen by expressing their components in terms of the *commuting* spinors

$$\hat{r}^{\pm} = \begin{pmatrix} \hat{r}_1^{\pm} \\ \hat{r}_2^{\pm} \end{pmatrix} \tag{5.5}$$

$$\begin{aligned} \hat{v}_0 &= -\frac{\alpha}{2}\hat{r}^{+T}\mathbb{I}\hat{r}^- \\ \hat{v}_1 &= \frac{\alpha}{2}\hat{r}^{+T}\sigma_3\hat{r}^- \end{aligned}$$

$$\hat{v}_2 = \frac{\alpha}{2}\hat{r}^{+T}\sigma_1\hat{r}^-$$

$$\hat{\hat{H}} = -i\frac{\alpha}{2}\hat{r}^{+T}\sigma_2\hat{r}^- \tag{5.6}$$

where the $\sigma_{1,2,3}$ are the usual Pauli matrices and, from (4.1)

$$[\hat{r}^+, \hat{r}^-] = 0$$

Note that the above vector $\hat{v}_a$ and $\hat{\hat{H}}$ require *both* the $\pm$ spinors (5.5). This is in contrast to the generators $\hat{j}_a^+$ and $\hat{j}_a^-$ (4.7-8) of the two commuting sl(2,$\mathbb{R}$) subalgebras (4.9).

The extended algebra of the *ten* $\hat{\hat{H}}, \hat{j}_a^\pm, \hat{v}_a, a = 0, 1, 2$ closes as follows

$$[\hat{j}_a^\pm, \hat{j}_b^\pm] = 2i\hbar\epsilon_{abc}\hat{j}^{c\pm}$$

$$[\hat{j}_a^+, \hat{j}_b^-] = 0 \tag{5.7}$$

$$[\hat{\hat{H}}, \hat{v}_a] = i\hbar\alpha\hat{P}_a = \frac{i\hbar}{2}(\hat{j}_a^+ - \hat{j}_a^-) \tag{5.8}$$

$$[\hat{\hat{H}}, \hat{j}_a^\pm] = \pm i\hbar\hat{v}_a \tag{5.9}$$

$$[\hat{v}_a, \hat{v}_b] = -\frac{i\hbar}{2}\epsilon_{abc}(\hat{j}^{c+} + \hat{j}^{c-}) \tag{5.10}$$

$$[\hat{j}_a^\pm, \hat{v}_b] = \mp i\hbar\eta_{ab}\hat{\hat{H}} + i\hbar\epsilon_{abc}\hat{v}^c \tag{5.11}$$

with the identities

$$\hat{v}^a\hat{j}_a^\pm = \hat{j}_a^\mp\hat{v}^a = \pm\frac{3i\hbar}{2}\hat{\hat{H}} \tag{5.12}$$

in addition to (4.11),(4.12) and (5.4), making a total of 6 identities.

The above 10-dimensional algebra is isomorphic to the Lie algebra of so$(2, 3)$, whose corresponding group is the conformal group of 3-dimensional Minkowski space. The dilatation $D$ is to be identified with - $\hat{\hat{H}}$, the translations with $\hat{P}_a^-$, and the conformal accelerations are denoted by $\hat{P}_a^+$, where

$$\hat{P}_a^\pm = \alpha\hat{P}_a \pm \hat{v}_a$$

## VI. THE QUANTUM MODULAR GROUP

The modular group acts classically on the torus modulus and momentum as

$$
\begin{aligned}
S &: m \to -m^{-1}, \qquad p \to \bar{m}^2 p \\
T &: m \to m+1, \qquad p \to p
\end{aligned}
\tag{6.1}
$$

and is equivalent to its action on the holonomy parameters

$$
\begin{aligned}
S &: r_1^{\pm} \to r_2^{\pm}, \qquad r_2^{\pm} \to -r_1^{\pm} \\
T &: r_1^{\pm} \to r_1^{\pm} + r_2^{\pm}, \qquad r_2^{\pm} \to r_2^{\pm},
\end{aligned}
\tag{6.2}
$$

Either (6.1) or (6.2) can be used to check that the Hamiltonian and Poisson brackets are invariant, and that the constants $C_1 - C_6$ transform as

$$
\begin{aligned}
S :&C_1 \to C_2, \quad C_2 \to C_1, \quad C_3 \to -C_3, \\
&C_4 \to C_5, \quad C_5 \to C_4, \quad C_6 \to -C_6 \\
T :&C_1 \to C_1 + C_2 + 2C_3, \quad C_2 \to C_2, \quad C_3 \to C_2 + C_3, \\
&C_4 \to C_4 - C_5 + C_6, \quad C_5 \to C_5, \quad C_6 \to C_6 - 2C_5
\end{aligned}
\tag{6.3}
$$

whereas $v_a$ and the $\mathrm{SO}(1,2)$ generators $j_a^{\pm}$ transform as

$$
\begin{aligned}
S :&j_0^{\pm} \to j_0^{\pm} \\
&j_1^{\pm} \to -j_1^{\pm} \\
&j_2^{\pm} \to -j_2^{\pm} \\
T :&j_0^{\pm} \to \frac{3}{2} j_0^{\pm} + \frac{1}{2} j_1^{\pm} - j_2^{\pm} \\
&j_1^{\pm} \to -\frac{1}{2} j_0^{\pm} + \frac{1}{2} j_1^{\pm} + j_2^{\pm} \\
&j_2^{\pm} \to -j_0^{\pm} - j_1^{\pm} + j_2^{\pm}
\end{aligned}
\tag{6.4}
$$

With the ordering of (4.2) (the only ambiguity), the quantum action of the modular group is the same as the classical one, with no $O(\hbar)$ corrections, and is generated by the $\mathrm{SO}(2,2)$ anti-de Sitter subgroup by conjugation with the operators $U_T$ and $U_S$ where

$$
U_T = \exp \frac{i}{2\hbar}(j_0^{\pm} + j_1^{\pm}) = \exp \mp \frac{i\alpha}{2\hbar} C_2^{\mp} = \exp \mp \frac{i\alpha}{2\hbar} (r_2^{\pm})^2 \tag{6.5}
$$

$$
U_S = \exp \frac{i\pi}{2\hbar} j_0^{\pm} = \exp \mp \frac{i\pi\alpha}{4\hbar}(C_1^{\mp} + C_2^{\mp}) = \exp \mp \frac{i\pi\alpha}{4\hbar}((r_1^{\pm})^2) + (r_2^{\pm})^2) \tag{6.6}
$$

The first of these (6.5) appeared in [7] in a different notation. The second (6.6) was calculated independently by one of us (J.E.N.) and S. J. Carlip. The

remaining generator $j_2^{\pm} = \pm C_3^{\mp}$ acts, with parameter $\epsilon$, on the holonomy parameters as

$$\exp(\frac{i\epsilon}{\hbar} j_2^{\pm})\ r_1^{\pm}\ \exp(-\frac{i\epsilon}{\hbar} j_2^{\pm}) = r_1^{\pm}\ \exp -\epsilon$$
$$\exp(\frac{i\epsilon}{\hbar} j_2^{\pm})\ r_2^{\pm}\ \exp(-\frac{i\epsilon}{\hbar} j_2^{\pm}) = r_2^{\pm}\ \exp \epsilon \tag{6.7}$$

so that the moduli and their momenta scale as

$$m \to m\ \exp -2\epsilon, p \to p\ \exp 2\epsilon \tag{6.8}$$

It can be checked that, using (6.2), the commutators (4.1) and therefore the quantum algebra of Section V, and the identities (4.11-12), (5.4) and (5.12) are invariant.

## VII. CONCLUSION

We have described several different (and possibly inequivalent) quantisations of 2+1 - dimensional gravity on $\mathbb{R} \times T^2$ in the presence of a negative cosmological constant. The direct ADM approach leads naturally to a well-defined Schrödinger dynamics (with positive definite, self-adjoint Hamiltonian operator) but does not succeed in exploiting, or even implementing, operator analogues of the classically independent conserved quantities arising from traces of holonomies. Perhaps this is not surprising since such a result (effectively a solution of the Heisenberg equations of motion) would encode within it the details of the spectrum of the Laplace-Beltrami operator on moduli space- a heretofore unsolved problem. For this same reason we suspect that the elegant explicit solution of a different formulation of the Heisenberg equations of motion given in [5] is not unitarily equivalent to the solution of the Schrödinger equation involving the square root of the Laplace-Beltrami operator (which was studied in detail in [2]). Indeed a representation for the fundamental quantised holonomy parameters which guarantees positivity of the Carlip-Nelson form (2.36 and 4.2) of the reduced Hamiltonian (2.18) does not seem to be known.

On the other hand if we set aside the above questions and quantise directly in terms of the holonomy parameters we can easily order the generators of the so$(2,2)$ algebra, or, more generally, those of the so$(2,3)$ algebra, so as to implement these algebras quantum mechanically. Within this same context we also formulate the

action of the quantum modular group as generated by a certain discrete subgroup of the SO$(2,2)$ anti-de Sitter group.

The conserved quantities $C_1 - C_6$ are shown explicitly to be global and time independent by expressing them in terms of the parameters of the traces of holonomies through a time dependent canonical transformation. Certain combinations of them satisfy the Lie algebra of the anti- de Sitter group SO$(2,2)$. Quantisation is straightforward in terms of the holonomy parameters. When the Hamiltonian is included three new (quantum) conserved quantities are found and the algebra extends to that of the conformal group SO$(2,3)$. The quantum modular group is generated by the anti- de Sitter subgroup, namely by these (quantised) conserved quantities. The quantum modular group appears as a discrete subgroup of the conformal group.

The group extension, that is, the Hamiltonian and the vectors $v_a$ act differently by mixing the two so$(1,2)$ algebras. The role of these operators and their action on the quantum states of the system is under investigation.

A related construction for zero cosmological constant using ADM variables can be found in [11]. There the constants found by one of us [8] are used as generators of isometries in the unreduced, ADM, Hamiltonian formalism.

For completeness we note that although we have only discussed the case of negative cosmological constant there would seem to be no obstruction to the discussion for $\Lambda$ positive or zero (see the discussion in [5]). For example, for $\Lambda > 0$, the parameters ${r_{1,2}}^{\pm}$ would be unchanged but the holonomies (2.24), the "shifted connections" (2.6), and in consequence, the corresponding sl$(2,\mathbf{C})$ generators ${j_a}^{\pm}, a = 0, 1, 2$ would be complex conjugates of each other rather than real and independent as here, for $\Lambda < 0$.

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